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# CHAOS AND FRACTALS: THEORY AND APPLICATIONS

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## Resumen

*La complejidad constituye una de las más importantes propiedades características de las distintas parcelas del conocimiento, constituyendo la Economía Dinámica Caótica un punto de partida que surgió precisamente con el objetivo de comprender, estructurar y explicar lo que realmente debemos entender como complejidad. En este artículo, tras una introducción sobre los conceptos y técnicas referentes a las matemáticas del caos y a la geometría fractal pasamos a analizar las posibilidades de aplicación de tan valiosos instrumentos a las diferentes áreas de la Ciencia, llevando acabo un conjunto de consideraciones de carácter general dentro de un amplio panorama, deteniéndonos a continuación en el estudio de dos cuestiones de especial importancia: la existencia de caos en series temporales, por una parte, con aplicaciones puntuales a los Mercados de Capitales durante la crisis recientemente padecida, utilizando los datos diarios del IBEX 35 en los años 2006-2013 y, por otra, las posibilidades que se dan en el amplio campo de la Ingeniería en conexión con importantes apartados de la Física fundamental.*

*Palabras claves: Complejidad, caos, fractales, no-linealidad, series temporales, mercados de capitales, ingeniería, física.*

## Abstract

*Complexity is one of the most important characteristic properties of the different fields of the knowledge, constituting the so called Chaotic Dynamic a relevant area that emerged with the objective of understanding, structuring and explaining in an endogenous way such complexity. In this paper, and after scanning the principal concepts and techniques about the mathematics of chaos and the fractal geometry, we analyze the possibilities of application of these valuable tools to the different areas of Science, carrying out a set of general considerations within a broad picture, stopping then in the study of two issues of special importance: the existence of chaos in time series, on one side, and the eventual chaotic behaviour in the Capital Markets during the recent period of crisis, considering the IBEX 35 daily data in the years 2006-2013, and for other, the possibilities that are given in the broad field of engineering in connection with physics.*

*Keywords: Complexity, chaos, fractals, non-linearity, time series, capital markets, engineering, physics.*

## Introduction

Since the last half of the 20th century the concepts and ideas of chaos and fractals have had developed a very important impact on a set of scientific disciplines, such as astrophysics, general relativity and cosmology, atomic and molecular dynamics, fluid dynamics, plasma physics, etc. But at the same time they have affected many other fields as well, which in principle seem are far away from physics, such as economics or ecology, which have also turned out to be fruitful environments where fractal structures appear naturally. We are going to approach how much significant has been the impact on different areas of Science, but before it is necessary to deal with the concepts, content and extent of chaos and fractals.[1]

Perhaps the most clear and ancient definition of chaos can be found in the words of Giordano Bruno written in his book published in 1583 in Venezia whit the title "De l´infinito universo mondi*": Now more than ever I perceive that a tiny error in the beginning causes a big difference and a serious deviation at the end; a single problem was multiplied gradually branching out into an infinite number of other, just as a root spreads in infinite branches and masses.* Also, centuries later, in 1908, appears the G. K, Chesterton´s celebrated novel "The Man Who was Thursday", and in the page 12 said: *Why do all the clerks and navvies in the railway trains look so sad and tired, so very sad and tired?. I will tell you. It is because they know that the train is going right. It is because they know that whatever place they have taken a ticket for, that place they will reach. It is because after they have passed Sloane Square they know that the next station must be Victoria, and nothing but Victoria. Oh, their wild rapture! Oh, their eyes like stars and their souls again in Eden, if the next station were unaccountably Baker Street.* These two relevant references are indeed very clear and rich in ideas about chaos, and it is an excellent starting point for our analysis that will be related, obviously, to complexity, given that it means diversity, creativeness, interactions at different levels and the possibility of emergent properties.

The paradigm of the complexity constitutes an obligatory reference to the scientific analysis on the verge of the XXI century, and this is especially true in the field of Economics, which is accustomed to navigate against the tide, and subject to a continuous dispersion, such as the one we had the occasion to emphasize in a paper published some years ago.[2] The appeal of the complexity, in the case of Economics, warrants its interpretation as a healthy exercise, worthy of defense

[1] AGUIRRE, Jacobo; VIANA, Ricardo L. and SANJUÁN, Miguel A.F. (2009): p.381.

[2] FERNÁNDEZ DÍAZ, Andrés (2000): pp. 39-44.

before the reductionist determinism professed by this science throughout its history, succumbing all too often to the aesthetic pleasure of simplicity. Should it not be clear, and with a view to taking a stand, it should be stated, with no undue delay, that determinism is pernicious: when we are distracted we deliberately skip the complexity.

We obviously do not seek the disqualification of determinism in radical support of decisive indeterminism, nor enter into the protracted and inconclusive polemics set forth all around it. What really and powerfully attracts the attention is the fact that a natural science, such as Physics, may have spectacularly advanced, which supposes its quantum revolution, through basis on the Heisenberg's uncertainty principle and the well- known interpretation of Copenhagen, due to Niels Bohr, in as much as, it reigns over a rigid determinism in the most complex sphere of the spirit, of the culture and society, or, tantamount to the same, of a social science such as Economics.

With a view to entrenching the question momentarily, and without hazard of remission to other more specific and detailed works on the matter, determinism could be understood as an abstraction and simplification, to make everyday complexity intelligible, considering, for its part, indeterminism as a consequence of our inability to explain complication, due to the fact that we do not avail of sufficient information. From this point of view, indeterminism would then be a clear consequence of complexity.

Below the broad parasol of complexity, we find ourselves in a fascinating world of concepts, terms and instruments, bustling, intertwined, and opening new horizons in almost all fields of knowledge. Dynamics, non-linearity, irregularity, order and chaos are only some of them, behaving as parts of an indivisible whole. To all this, it is necessary to add a type of strict definition of complexity, which is habitually used in the more recent works of specialists. In them, the term *complex* is used, to refer to those cases in which dynamic long-term behaviour is more complicated than a fixed point, a cycle limit, or a torus; or tantamount to the same, when chaotic behaviour is produced.

It is very important to highlight  that in the study of chaos as a subset of complexity is very available don't forget that the major problem in time series research is the difficulty of distinguishing between deterministic chaos and a purely random process, taking into consideration that the most important characteristic of chaotic dynamical systems is their short-term predictability. Chaos is at the same time disorder and determinism. Chaos, in principle, due that is apparently disordered, make non predictable its evolution. But, on the other hand, being deterministic, and governed by systems of non-linear equations, it should be possible to predict and control once you know the mathematical relationships of the variables that influence it. As said Henri Poincare is much better to look farther without having certainty, that don't look anything at all. Because of this we must to undertake the analysis of the main concepts, techniques and mathematics of chaos, that is, of all the weapons we need to know and to deal with an irregular and complex reality, as already we have pointed out.

Before of going on, however, it is necessary to know that complexity and chaos are intimately related to the concept of emergence. What does emergence means?. Taking into account the evolution approach, we can think that emergent evolution may be interpreted as an incessant flow of creative novelty, which implicate a special conception of the whole and the parts, farther away the simple and lineal idea of an additive process.[3] In reality there is a process of emergence when the behaviour of the overall system cannot be obtained by summing the behaviours of its constituent parts. That is, the whole is indeed more than the sum of its parts.

The habitual definition of chaos, which hallmark is the sensitive dependence on initial conditions, implies that there is no information within chaos, and it has neither form nor structure. For us, chaos may be complex and appear to be non-deterministic, but hidden within it is a wealth of information. If in an emergent phenomenon there is also some hidden information, given that, as we have seen, the whole became something more than the sum of its components, seems clear, first, the closed relation between chaos and emergence, and secondly, the help that the last one can render to the predictability of chaos. We must remember this very important consideration in the conclusions of this work.

Finally it is important to highlight in this introduction that we shall consider the applications of complexity, chaos and fractals to different areas of science, especially to economics and in some types of engineering.

## Mathematics of Chaos and Fractal Geometry

The characteristics of irregularity and non-linearity are, among others, derived from the complexity of different branches of science, which oblige, as stated at the beginning, the utilization of concepts and new instruments especially conceived to

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[3] FERNÁNDEZ DÍAZ, Andrés (1999$_a$): pp. 139-145.

face challenges. Amongst them the Theory of Catastrophes and very especially, the Mathematics of Chaos stand out.

The majority of authors coincide in as much as the Theory of Catastrophes and the Mathematics of Chaos can be considered as two approaches to a general theory of dynamics of discontinuities. Both have in common as a base the idea of a splitting or halving of the equilibrium at critical points, just as the fact that functional relation are, with greater frequency, of the nonlinear type. But they differ in as much as some discontinuities are set forth on a great scale: the Theory of Catastrophes, and others on a small scale: the Mathematics of Chaos. The Theory of Catastrophes is therefore a special case of the bifurcation theory accredited originally to Poincaré, which contemplates the world as essentially uniform and stable yet subject to sudden changes, the unexpected, or discontinuities on a grand scale which are produced in certain variables of state.

It is well known that the starting point of the Theory of Catastrophes can be found in the works of René Thom and Christopher Zeeman, at the end of the sixties and the beginning of the seventies. On other occasions we have, and at certain length, taken to hand this new mathematical method, in order to describe the evolution of forms in nature, by hazarding even some economic applications, and concretely, the problem of stagflation[4]. We shall not go into this any further. Instead, we shall center our attention on Chaos and its measurement, which has greater relevancy for the purposes of our analysis.

It is often said that chaos is a ubiquitous phenomenon which is produced everywhere and can be observed in all fields of Science. Thus we find chaotic systems in the Hamiltonians, in the three bodies of celestial mechanics, in the physics of fluids, in lasers, in particle accelerators, in biological systems, in chemical reactions, and as we shall soon see, in no small part of the behavioural forms within the field of economics.

Chaos can be located through the function of ***strange attractors***, by following bifurcation diagrams, or by analyzing the intricate profile of figures of fractal geometry. It should not be forgotten, in this respect, that, as Giambattista Vico said, chaos is *the raw material of natural things that, shapeless, is thirsty for form, and devours all*. We know that the essential geometrics of chaos consist of stretching and bending, as pointed out by Stephen Smale in his topologic transformations. In

[4] FERNÁNDEZ DÍAZ, Andrés (1999b): pp. 45-48.

effect, the irregularity of movement is produced by a mechanism which is broken down into two actions. On the one hand, the spatial phase is stretched, by separating trajectories, and then it doubles back onto itself.

The exponents of Lyapunov serve to explain the first part of this process, on giving a measurement on the exponential separation of two adjacent trajectories. We can study local instability of a discrete system $x_{n+1} = f(x_n)$ in the Lyapunov sense measuring how two adjacent points separate with the iterated application of the function, that is,

$$|f^n(\mathrm{x}_0 + \epsilon) - f^n(\mathrm{x}_0)| = \epsilon^{n\lambda(x_0)} \tag{1}$$

The limit

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \left|\frac{f^n(x_0+\epsilon) - f^n(x_0)}{\epsilon}\right| \tag{2}$$

or also:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{df^n(x_0)}{dx_0}\right| \tag{3}$$

are both (2) and (3) path expressions of the Lyapunov exponent. In a general n-dimensional system there will be *n* Lyapunov exponents, showing each of them the average rate of expansion or contraction of the phase space in each of its *n* direction under the action of the dynamical system.

These Lyapunov exponents can be used to distinguish simple dynamic attractors from complex dynamics attractors, that is, those that we could call existing traditional attractors until the Lorenz contribution in 1963, *fixed points*, *limit cycles* and *quasi-periodic torus,* from chaotic or strange attractors.

We can say that the dynamics inside fixed points, limit cycles or torus is a simple dynamics in the sense that two orbits that start arbitrary near each other, always remain adjacent, thus providing a guaranteed predictability on a long term basis. However, inside the strange or chaotic attractor the dynamical system is complex in the sense that although bounded inside the attractor is also locally instable, with recurrent but aperiodic cycles, and with sensitive dependence to initial conditions making predictions difficult beyond the very short term. Therefore “simple dynamics” and “complex dynamics”, can be detected by means of the Lyapunov exponent. More concretely, a positive Lyapunov exponent, i.e. local instability, is a necessary condition for the existence of chaos.

If we are situated in a dimension, we have only stable fixed points with the exponent $\lambda$ negative. In the case of two dimensions, the attractors would be fixed points, with negative exponents, and limit cycles with $(\lambda_1, \lambda_2) = (-, 0)$.

In three dimensions one would have:

$(\lambda_1, \lambda_2, \lambda_3) = (-,-,-)$ → (stable fixed point)

$(\lambda_1, \lambda_2, \lambda_3) = (0,-,-)$ → (stable limite cycles))

$(\lambda_1, \lambda_2, \lambda_3) = (-,0,0)$ → (stable torus)

$(\lambda_1, \lambda_2, \lambda_3) = (+,0,-)$ → (strange attractor)

There are many examples of strange attractors in specialized literature, beginning with Stephen Smale, who supplies through a topological transformation a base for the understanding of chaotic properties of dynamic systems. Among the stated attractors, the Lorenz attractor stands out, which takes on the form known as the wings of the butterfly, and in the one that borders on an important problem for meteorology, the one about atmospheric convection, consisting of the evolution of a layer of fluid heated from below. To this emblematic attractor, other examples should be added, such as the not less known one of Rössler, and the one by Hénon, which has an elegant structure and is quite complex. But we cannot speak of strange attractors without entering into the attractive and fascinating field of fractals.

The *chaotic attractors* are fractals. Fractals are geometric objects which have a beautiful microscopic structure that have been developed within the framework of a new form of the geometry of nature or of the complexity created by Benoît Mandelbrot, who in 1975 published his famous work called ***The Fractal Geometry of Nature***. He was educated at "École Normale" and the "École Polytechnique" and, with his original formulation, intended to confront the unbounded formality of the Bourbaki group, thereby reestablishing the image and prestige of Henri Poincaré.

Some authors affirm very often that scientists know a fractal when they see one, but there is not universally accepted definition. It is generally acknowledge that fractals have some or all of the following properties: complicated structure at a wide range of length scales, repetition of structures at different length scales, and a fractal dimension that is not an integer.[5]

It is necessary to point out that fractality is ubiquitous in nature, as has been observed since the relation between fractality and nonlinear dynamics was established. In the context of dissipative systems, examples of fractal behaviour are numerous, noting the appearance of fractal basins in a wide variety of nonlinear oscillators such as the Duffing one or the forced damped pendulum, and multispecies competition or predator-prey models.[6]

The fractal concept involves a new idea of dimension beyond the Euclidian one, given that it dealt with the fractal or intermediate dimensions that come, in essence, into prominence if it is considered that the system does not occupy all the space that corresponds to its Euclidean dimension. In effect, the fact that the dimension may be inferior to the number of parameters or degrees of freedom, necessary to completely specify the state of the system considered, signifies that it does neither exploit all the possibilities, nor all the states theoretically possible.

The fractal should be understood as a geometric form which remains unaltered, whatever the increase in which it is observed. It could be said that, within reasonable, the fractal has the same structure on all scales, the contrary to what occurs in the phenomenon of the renormalization in which the figures notably alter when they are modified or vary.

The fundamental problem of fractals lies in knowing their dimension, which does not necessarily need to be an integer, as we have already noted. Originally, the numeric measurement of the degree of rigorousness was denominated as the Hausdorff-Besicovitch dimension; today it is called the fractal dimension. In general terms, the dimension is a measure of the occupation of space by a geometric object. Normal non-fractal objects have a Hausdorff dimension equal to its topological dimension. However, fractals have a Hausdorff dimension strictly greater than its topological dimension.

[5] ALLIGOOD, Kathleen T.; SAUER, Tim D.; YORKE, James A. (1997): pp- 149-150.
[6] AGUIRRE, Jacobo; VIANA, Ricardo L.; SANJUÁN, Miguel A.F. (2009): p. 334.

One of the methods employed for carrying out the measurement of fractals is based on the concept of homotecia in Euclidian geometry, which also allows the calculation of fractal dimension through basis on the concept of capacity:

$$D_0(S) = \lim_{\epsilon \to 0} \frac{\log M(\epsilon)}{\log\left(\frac{1}{n}\right)} \tag{4}$$

where *S* is a subset of the *n*-dimensional space, and *M*(ε) the minimum number of ε-side n-dimensional cubes necessary for covering such a subset. For small values of ε, the implicit definition shown in (4) means that:

$$M(\epsilon) \propto K \cdot \epsilon^{-D_0} \tag{5}$$

The capacity of a point, a line or an area in the bi-dimensional space, takes the values 0, 1 and 2 respectively. That is, if we take the cubes of the side ε, the number required to cover the point would be proportional to $1/\varepsilon^0$, to cover the line to $1/\varepsilon^1$, and to cover the surface to $1/\varepsilon^2$. The dimension of fractal sets, as mentioned previously, is strictly greater that this Euclidean capacity. Thus, the Kotch curve and the Cantor set, which constitute typical examples of fractals, have fractal dimension greater that one and 0, respectively:

$$d = \frac{\log 4}{\log 3} = 1.2619 \qquad and \qquad d = \frac{\log 2}{\log 3} = 0.6309 \tag{6}$$

We must know that the middle-thirds Cantor set (cif. Fig. 2) is the simplest mathematical fractal that one cam study. Likewise an idealized model of a fractal coastline, the Koch curve, is made by the Cantor process of fulling up middle thirds to make tents (or roof-tops of houses), as is indicate d in Fig. 1. Let us now see both graphical representations.[7]

[7] In addition we could mention the Sierpinski carpet as another of the basic or elemental fractals. See: McCAULEY, Joseph L. (1993): pp. 129-132.

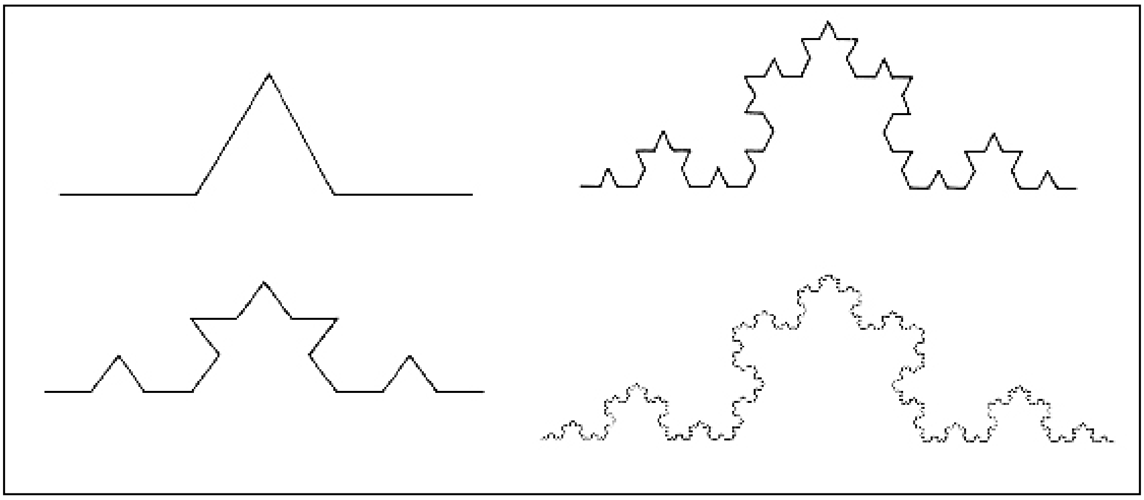

Figure 1. Kotch curve

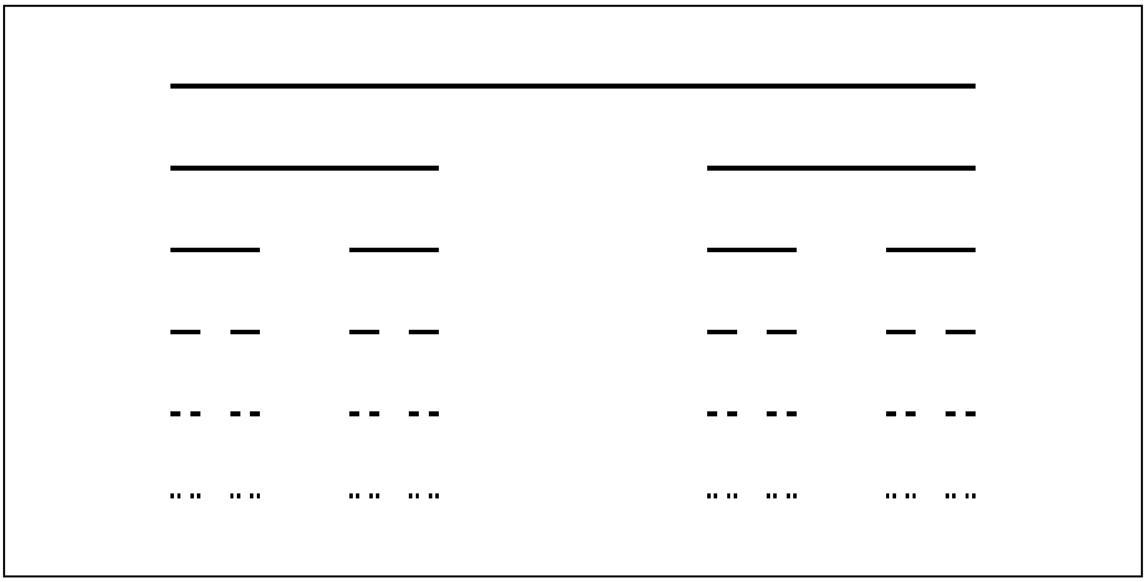

Figure 2. Cantor Set

Lyapunov exponents and fractal dimension are the two main instruments of the Mathematics of Chaos. Another key concept in the analysis of chaos and complex dynamics is bifurcation, which could be defined as a doubling of the period of the attractors as some parameter of the system changes. With the analysis of the occurrence of this bifurcation, that is, the comparative dynamics of the system when the value of a parameter is changed, it is possible to explore the different types of attractors that the model can achieve. And the way in which the changes in the parameters move the system from one attractor to another is precisely through the bifurcation of the period.

Without forgetting the contributions of Yorke and May, and of course, the clear and decisive inspiration of Smale, the best known and illustrative analysis of comparative dynamics is the *Feigenbaum bifurcation tree*.[8] Consider, for example, the logistic map

$$x_{t+1} = \mu x_t(1 - x_t) \tag{7}$$

where $\mu$ is a constant situated at the interval [0,4]. When we iterate the map from an arbitrary initial condition, the attractor of this discrete dynamic system is obtained depending of the values assigned to $\mu$:

| | | |
|---|---|---|
| If $0 \leq \mu < 3$ | $\rightarrow$ | a sole stable fixed point |
| $\mu = 3$ | $\rightarrow$ | a marginally stable fixed point |
| $\mu > 3$ | $\rightarrow$ | a fixed point becomes unstable |
| $\mu = 3.2$ | $\rightarrow$ | second period cycle |
| $\mu = 3.5$ | $\rightarrow$ | fourth period cycle |
| $\mu = 3.56$ | $\rightarrow$ | the period has doubled to eight |
| $\mu = 3.567$ | $\rightarrow$ | the period has doubled to sixteen |
| $\mu = 3.58$ | $\rightarrow$ | the cascade of duplications is so rapid that the logistic map becomes chaotic |

and the graph of bifurcation would be the following:

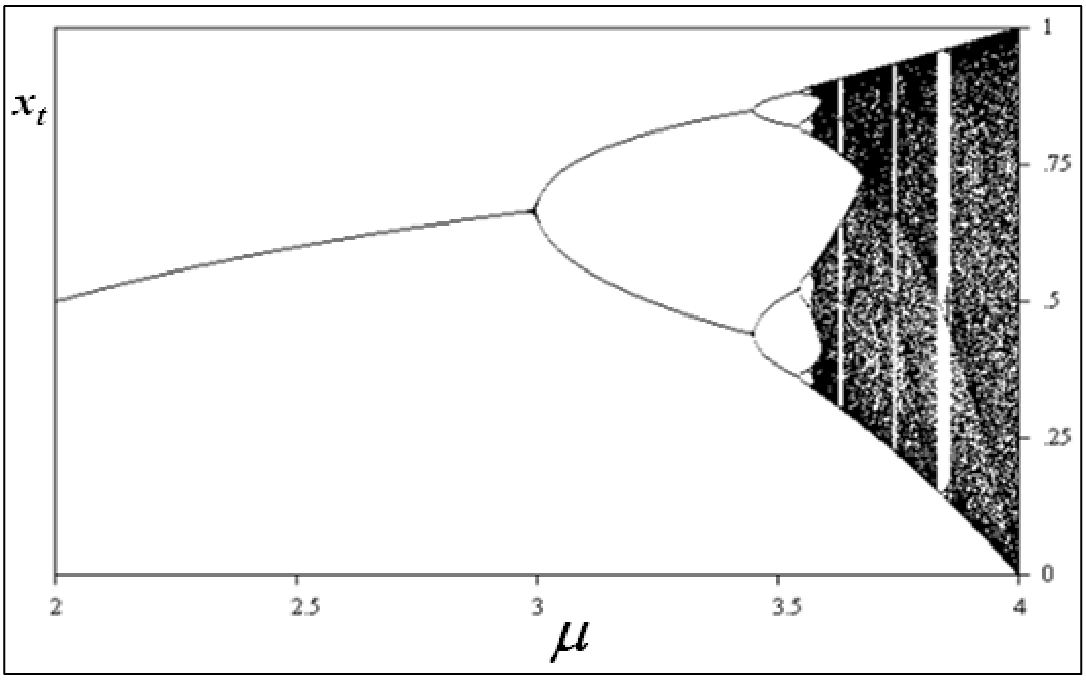


Figure 3. Feigenbaum Tree

8 AlLIGOOD, Kathleen T.; SAUER, Tim D.; YORKE, James A, (1997): pp. 448-460.

In this way, by advancing through the duplication or the bifurcation of the period of the attractor, the Feigenbaum Tree is obtained (*figure 3*).The scale factor of the tree branches tends towards a universal Feigenbaum constant, which is equal to 4.6692, and is maintained even if we use another application. In effect, the relation

$$\delta_i = \frac{\mu_i - \mu_{i+1}}{\mu_{i+1} - \mu_{i+2}} \qquad (8)$$

converges towards this universal number, being therefore $\delta_\infty = 4.66922$.

It should be added that, in the zone of chaos, small windows or oases of order and stability can be found in the middle of disorder, illustrated in the habitual graphic representations by means of clear spots in foggy and obscure areas.

## Chaos in time series: the tests to detect it

There are abundant contributions in the scientific literature showing that it is possible to generalize the traditional theoretical models to show chaotic behaviour under plausible assumptions. However, while there is not a great difficulty to design theoretical models in regime of chaotic behaviour, there is no clear evidence that the correspondent time series behave chaotically.

In fact, the major advances in the application of chaos theory in each case deal with the tools to detect if the underlying true time series generation process is really a chaotic dynamic system. The detection of chaos in the underlying dynamics of a time series is divided into several stages. The first step in detecting chaotic behaviour from a time series is to find evidence of nonlinear time dependence in the underlying dynamics of the system. And for that, the most widely used tests in the diverse fields are the BDS test and the Hurst exponent, which are two techniques to contrast the existence of time dependence, linear or non-linear.[9]

To detect nonlinear dependence the Brock test is used. This test consists on filtering the time series by a general auto-regression model with a range large enough to ensure that any linear dependence has been completely removed. If,

[9] BROCK, W.A.; DECHERT, W.D.; SCHEINKMAN, J.A. and LeBARON, B. (1996): pp.197-235.
MANDELBROT, Benoît. (1972): pp. 259-290

despite the linear filtering, Hurst and BDS tests continue to show evidence of time dependence, and then it must be nonlinear.

Once detected the non-linear dependence, the next step is to estimate both the Lyapunov exponent and the fractal dimension of the attractor of the underlying system generating the time series in order to test if that dynamical system presents chaotic behaviour. The main limitation of this approach is that this system is unknown. It is for this reason that a previous step (prior to the estimation of Lyapunov exponents and fractal dimension) is the recovery or reconstruction of the attractor but maintaining the qualitative properties of the underlying unknown dynamical system generating the time series.

A commonly used method of reconstruction of the attractor is the *lag method*. This method is based on the *embedding* theorem of Takens (1985), that establishes that, under certain conditions, though it will not be possible to reconstruct the orbit of the dynamical system in the original phases space, it is possible to obtain an approximation of it that result equivalent in a topological sense (equivalence in the dynamic and geometric properties), and that permit to extract all the relevant information about the unknown underlying dynamical system that generates the time series. Once we have reconstructed the attractor from the time series, we can proceed now to estimate the fractal dimension and Lyapunov exponents to detect chaotic behaviour.

The Fractal dimension has a metric character, but using alternative measurement concepts to the traditional length, area or volume, and is usually calculated using covering formed by hyper-cubes or boxes. This method for calculating the fractal dimension is, however, little operational when working with embedding dimensions higher than two and when using time series contaminated by purely random noise, and for that reason alternative methods have been developed. Among them are the methods that use the ergodic theory to calculate a probabilistic measure of the attractor, the frequency with which the orbit visits the different parts of the attractor. Among these probabilistic dimensions, the more generally used is the correlation dimension that we explain shortly. The method consist, fundamentally, in centering an hypersphere in a point of the phase space making growth the radio r of the sphere until that all the points remain into it.

We can write the correlation function between two points for a small r in this way:

$$C_m(r) = r^{D_c} \tag{9}$$

where r is the sphere radio, m the embedding dimension, and $D_c$ the correlation dimension.

Then, taking logarithm we should have

$$l_n\, C_m\, (r) = D_c\, l_n\, r$$

$$D_c = \frac{\ln C_m\ (r)}{\ln r} \tag{10}$$

That is the correlation dimension, been demonstrated by Grassberger and Procaccia that its values are near to those of the capacity dimension without exceed them. That is

$$D_c \leq D$$

If the correlation dimensions growth with m, that is, with the embedding dimension, the process will be stochastic, and if is independent of m, the process will be deterministic.[10]

The fractal dimension, in his stead, provides a measure of the complexity of the attractor. However, the estimation of fractal dimension from a time series cannot be taken as a sufficient test for the detection of deterministic chaos. This is because firstly, it is only possible to obtain rough estimates of the true fractal dimension of an unknown dynamic system, and therefore, it is very risky to assure when this approximation is an integer or fractional. Second, because when working with economic time series, we must accept the fact that in the series there is always some random component, so that the estimate of the fractal dimension will be always biased upwards, making it difficult the detection of low-dimensional chaotic behaviour.

[10] FERNÁNDEZ DÍAZ, Andrés (1994): pp. 150-151.
GRASSBERGER, P. and PROCACCIA, I. (1983): pp. 189-208.

Therefore, the estimation of fractal dimension in the search for a non-integer dimension and not very high, must be taken as a supplement to other techniques for the detection of chaos, especially, the spectrum of Lyapunov exponents. Recall that in dissipative systems, the presence of a positive Lyapunov exponent is indicative of sensitive dependence to initial conditions, and then it is the sufficient condition to chaotic dynamics to exist.

There are several algorithms to measure the Lyapunov exponents of the underlying time series generation process. The *Wolf et al (1985) direct algorithm* may not provide a correct characterization of Lyapunov exponents of a time series with limited number of observations. Furthermore, the performance of this direct algorithm is very sensitive to the degree of noise in the data. For these reason in Economics, with time series characterized by short sample and error measured, we use indirect methods that use regressions method to estimate the underlying derivative in (4).

These regression methods to estimate indirectly Lyapunov exponents assume the existence of an unobserved dynamic model that may be chaotic

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-d}) + e_t \qquad (11)$$

where $t$ =1, 2,...$N$, and $\{e_t\}$ a sequence of *iid* random variables.

This model may be expressed in terms of a state vector $\vec{\boldsymbol{X}}_t$ an error vector $\boldsymbol{\varepsilon}_t$ , and a function $F$: $\mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\vec{\boldsymbol{X}}_t = \boldsymbol{F}(\vec{\boldsymbol{X}}_{t-1}) + \boldsymbol{\varepsilon}_t \qquad (12)$$

It is then possible to estimate Lyapunov exponents from the Jacobians of the map, that is, based on nonlinear regression estimates of *f* and *F* in the respective equations. There are different methods for estimating the map, but the NEGM and the NETLE methods, that take advantage of the use of multilayer feed-forward neural networks models, has been revealed as especially adequate for use with noisy data as well as with limited number of observations.[11]

[11] McCAFFREY, D.F.; ELLNER, S.; GALLANT, A.F.R. and NYCHKA, D.W.(1992): pp.682-695. GENCAY, R. and DECHERT, W.D. (1992): pp. 142-157.

Other method to detect chaos is, as we mentioned above, the analysis R/S, where R and S express respectively the range and the standard deviation of the observations (n) of a time series. We can calculate the regression line between log R/S and log n, whose slop H is the Hurst´s exponent.

In turn, a measure of correlation in function of H would be

$$C = 2^{(2H - 1)} - 1$$

At the same time it is necessary to emphasize that

$$\alpha = \frac{1}{H}$$

is a measure of the fractal dimension.

In this context we have:

$H = 0,50 \quad \rightarrow \quad C = 0 \quad$ (White noise)

$0 < H < 0,50 \quad \rightarrow \quad C < 0 \quad$ (Pink noise)

$0,50 < H < 1 \quad \rightarrow \quad C > 1 \quad$ (Black noise)

With white noise there would be no correlation; with pink noise would be an antipersistent or ergodic series, and with black noise the series would be persistent. Besides, we must say that series of this last type are fractals because they can be described with fractional Brownian motion. Its dimension, as we have just seen, is the inverse of H.

In the statistical observations in the Nile floods conducted over a period of time by the hydrologist Harold Edwin Hurst, it was concluded that the equilibrium position, or in other words, the optimum dam sizing, could be reached for a value of H around 0,67.

Persistent series can be interpreted as following a random biased walk, and the importance of bias depends on the degree to which H is above 0,50. On the other hand, this type of series, as we have said, is fractal, and given that the inverse of H is a measure of the fractal dimension, we can know with precision the dimensions of the fractals in the respective series.

## Controlling chaos

Before to conclude this first part of our work, it is necessary a short reference to the control of chaos, a subject that in the last years call the attention of the researchers about chaos. In this sense may be interesting to highlight that there is a new control method known as *partial control* that have been successfully applied to several important dynamical systems, such as the Henon map, the tent map, the Duffing oscillator and in the 3D Lorenz map, among others. At present, Levi, Sabuco and Sanjuán have published a paper considering, in the field of economics, the possibility of avoiding market collapse with partial control method.[12.] If we analyze the dynamics after a boundary crisis, we can find that the system possesses a transient chaotic behaviour in a bounded region in phase space previous to a situation in which the trajectory escapes towards an attractor outside this region. When the dynamics is affected by noise, somehow it might help the trajectory to escape from the region. In those circumstances the goal of the partial control is to apply a method that allows one to prevent the escapes of the trajectories to the external attractors, keeping the trajectories in the chaotic region forever.

In the theory of chaos the main objective of classical control methods has been to lock the dynamics into a specific steady state or periodic orbit, but there have appeared many situations where chaos could be a powerful attribute. So, in mechanics, for instance, chaos helps prevent undesirable resonance, and in engineering the thermal pulse combustor is more efficient in the chaotic regime, happening something similar in biology where the disappearance of chaos may be a signal of pathological behaviour. It therefore seems clear that in all these cases the existence of chaos is desirable as a status that should be preserved by preventing trajectories from escaping to an external attractor.

The authors mentioned in the last reference have applied the partial control method to the Lorenz system, needing for it a transient chaotic system with escapes, the knowledge of the upper bound of the disturbances and an upper bound control high enough to find a safe set with the Sculpting Algorithm. Those conditions are rather general, because in reality the controller only needs to know which the state of the system is and which the safe set is.

[12] LEVI, Asaf; SABUCO, Juan; SANJUÁN, Miguel A.F. (2017): pp. 1-7.
CAPEÁNS, Rubán; SABUCO, Juan; SANJUÁN, Miguel A.F. and YORKE, James (2016): pp.1-5.

## Some applications

Chaos is ubiquitous, and because of this it is not surprising that we find it in the different fields of knowledge. Among those fields in which we can analyse the role that the existence of chaos plays, we can mention the following:

-Applications in physical sciences

-Applications to biology

-Applications to physiology and medicine

-Atmospheric flight dynamics and chaos

-Application in naval engineering

-Applications in economics

Some of these branches of science are studied together, giving rise to a new type of application. Indeed, and as highlighted in the Review Nature in the year 2005, there is a new line of research, named “synthetic biology” that integrates different scientific areas, just as the non-linear dynamic, the physic of complex systems, the engineering and the molecular biology. This emergent field of research is intrinsically interdisciplinary and constitutes an advance to take into account for the next years in our work about the application of chaos and complexity theories. The studies of complex networks in physics, mathematics, economics and other sciences, is also at present undergoing a very important development, which represents new possibilities in the task that we are tackling.

We will now make a few brief remarks on the first four types of applications of chaos and fractals, with a greater focus on the two remaining types, namely, the naval engineering, on the one hand, and the economic science, for other.

a) Application in physics

Entropy is one of the most important concepts in Thermodynamics, or with other words, the most influential concept to arise from statistical mechanics, and has three related interpretation: entropy measures the disorder in a system; entropy measures our ignorance about a system; entropy measures the irreversible changes in a system. From the point of view of chaos the first two interpretations are of great importance, since they refer to the disorder, and to uncertainty and information.

In the evolution of the notion of entropy we have the definition of Rudolf Clausius in the context of the thermodynamic cycle of Sadi Carnot, and the concept of entropy information theory introduced by Claude Shannon. The first of the authors tell us that in isolated system there is a propriety named entropy that always growth

$$S(A) = \int_0^T \frac{dQ}{T} \geq 0$$

this is, for an isolated system the entropy is a growing function in the whole real process. Only if the process is irreversible, the entropy remains constant.

Respect to the Shannon contribution, in a probabilistic sense, the entropy is given by a function of the type

$$H(\xi) = H\left[P(X_1)..P(X_n)\right] = -\sum_{i=1}^{n} P(X_i)\log P(X_i)$$

that *ex–ante* gives us a measure of uncertainty and *ex–post* a measure of the information.

Within the field of physics has a special relevance, in the context we are considering, the Heisenberg´s uncertainly principle, decisive for overcoming the controversial determinism-indeterminism, to which we referred in the introduction. We could address some issues in such important branches as the quantum mechanics and particle physics, among others, but for obvious reasons we have to move on. It should pointed out, however, that we could deal here with the physics of the atmosphere and with processes such as the Bénard-Rayleigh convection, but we will leave it for when we stop in the field of naval engineering.

b) Application to biology

Biological systems appear as a paradigm of complexity and organization, given the high number of components that integrate it, on the one hand, and taking into account the special characteristics that binds to these components, both at the level spatial and temporal. It should be noted that it is common to find non-linear behaviour in biological systems, with increasing number of such systems in which there is chaos is a dynamic manifestation of non-linearity.

Let us not forget the first experimental system where chaos was detected was the enzymatic reaction of peroxidase, consisting of the reduction of one molecule of oxygen in water, acting the NADH as an electron donor:

$$O_2 + 2NADH + 2H \rightarrow 2H_2O + 2NAD$$

The system is kept open by the constant injection of oxygen and NADH into the enzymatic reaction medium, which proceeds with constant agitation. For different values of enzymatic concentration, the system shows different dynamic behaviours, among which it has been possible to detect *bistability, sustained oscillations, birritmicidad, oscillations to bursts and chaos.*[13]

We could go on with other illustrative examples, as that one of excitable systems where chaos is detected, both experimentally and from models, grouping a wide range of systems including unicellular organisms, endocrine organs of vertebrates and, of course, the nervous systems, so much of vertebrates as of invertebrates. Within the field of biology the chaotic dynamics in the behaviour of populations and ecosystem in general also has been described in models following the logic of the Lotka-Volterra equations or the dam and the predator.

In the framework of chaos research and its applications takes great importance and has many uses the known as *power laws* that are regularities that can be found, for instance, in many financial series or in the very interesting chapter of urban dynamics. With regard to the subject we are now discussing, it can be said that very often power laws have be identified in biology, including microbiology, developmental and evolutionary biology, ecology and paleontology. They show up as neat power laws valid over many orders of magnitude, as concatenated sets of power laws that describe hierarchical behaviour, or as trends in data for which there is little intuition.[14]

c) Application to physiology and medicine

Since the mid-eighties of the last century some well-known specialists have been working and exploring the application of non-linear dynamics, chaos theory and fractals to physiology and medicine. One of these specialists, A. L. Goldberger, professor in the Harvard Medical School, proposes the following practical application of chaos theory to cardiology:[15]

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[13] MONTERO, Francisco y MORÁN, Federico (1992): pp. 432-444.
[14] PÉREZ MERCADER, Juan (2001): pp. 354-358.
[15] GOLDBERGER, Ary L. (1992): PP. 321-329.

-Measuring loss of sinus rhythm interbeat interval complexity prior to cardiac arrest or other complications
-Detecting periodic heart rate (sinus rhythm or ectopic beat) dynamics prior to cardiac arrest or other complications
-Measuring loss of complexity of cardiovascular dynamics with aging
-Measuring loss of heart rate complexity with drug toxicity
-Detecting alternans phenomena before electrical or mechanical instability
-Detecting cyclic coronary blood flow with impending myocardial infarction

The study of these six points in the area of cardiology constitutes an excellent example to know the possibilities of applying chaos theory in the broad case of physiology and medicine. In this respect it is important to note that physiology may prove to be one of the richest laboratories for the study of chaos and fractals as well as other types of nonlinear dynamics. But it is necessary to develop a better understanding of how developmental processes lead to the construction of fractal architectures and how dynamic processes in the body generate apparent chaos. In this sense it has to be said that more recent studies of fractals and chaos in physiology may provide more sensitive ways to characterize dysfunction resulting from aging, disease and drug toxicity.[16]

We can consider the existence of chaos in brain activity as another example in the area of physiology and medicine. It is well known that the human brain contains approximately $10^{12}$ neurons, which are the functional cellular units of the nervous system. These cells are specialized in receiving information, making decisions and transmitting signals to other neurons or effector cells, such as muscle or glandular cells. Given that all this constitutes a complex system, it is logical to think that chaos can exist in the neural organization of superior animal, and in fact this chaotic dynamics has been showed on many levels, from the EEG analysis to the results obtained from the dynamics of neural network models.

Among others interesting examples we must to mention the synchronization and rhythmic processes in physiology, given that complex bodily rhythms are ubiquitous in living organisms. Such rhythms arise from stochastic, nonlinear biological mechanisms interacting with a fluctuating environment, and as a consequence the disease often leads to alterations from normal to pathological rhythm, something very important to take into account in the medical applications.[17]

[16] GOLDBERGER, Ary L.; RIGNEY, David R. and WEST, Bruce J. (1990): p. 49.
[17] GLASS, Leon (2001): pp. 277-283.

d) Atmospheric flight dynamics and chaos

In the main books and articles about this subject is considered that atmospheric flight dynamics *is the study of two interacting dynamical systems: the fluid system and the vehicle itself.* At the beginning modelling of the aerodynamic forces and moments acting on the vehicle was strictly linear, but with the wide development on nonlinear dynamic systems and the role of bifurcation and chaos in the late $1970_s$ and early $1980_s$ began a new and active area of research. Effectively, and coinciding with the previous definition, it is known that the most advance technology aircraft may be described as an integrated system comprising airframe, propulsion, flights control, and so on, and it is convenient to utilise powerful computational systems engineering tools to analyse and describe its flight dynamics.[18]

In the modern conception of space, within of micro-physics, the continent and the content are considered as inseparable, or to be more precise, in topological terms could be interpreted as *the same thing.* With other words, we can say that there is a *full interaction* between atmospheric flows (continent) and the aircraft (content). (See APPEDIX A in this work). This kind of indistinguishable mixing of atmospheric flows and aircraft implies the possibility of existence of chaos in the evolution of the system, and in fact since the early $1970_s$ the introduction of the concept and techniques of chaos and strange attractors provided different methods for characterizing the flight dynamics.

The dimension of the attracting subspace is one of the most important characteristic and there are several methods for determining this dimension. One of them is the Kaplan-Yorke conjecture with the Lyapunov exponents, a method that only provides an upper bound and is not easy to apply to experimental data where the Lyapunov exponents must be constructed. Several methods for determining the dimension of system that are readily applicable to experimental data also have come into use. Among these methods we can highlight the box-counting of Mandelbrot, the nearest-neighbour method of Pettis et al., and the correlation method of Grassberger and Procaccia that was discussed earlier in detail in the theoretical part of this article.

As the outstanding researchers who have dealt with this field of applications of chaos theory claim, the impact of chaotic behaviour on the concrete area of atmospheric flight dynamics has not been fully explored, considering that the proper

[18] CHAPMAN, Gary T.: YATES, Leslie A.: SZADY, Michael J. (1992): pp. 87-90.
COOK, Michael V. (2007): pp. 1-11.

modelling of complete aerodynamic characteristics is the key issue for successful studies about this matter.

e) Application in naval engineering

In the analysis of the applications of chaos theory to naval engineering it is necessary to take into account three fundamental pillars or sections that serve as reference or axes that define the space in which we can raise the set of questions and problems concerning the subject object of our attention. These pillars or sections are the following:

-Chaos and atmospheric flows

-Chaos in fluid flows

-Ship floatability and stability

Let us start with the first axis or coordinate of the three we just mentioned recovering the well-known meteorologist E.N. Lorenz, who was mainly concerned with the problem of weather forecasting. It is necessary point out that part of the turbulent motion of the atmosphere is caused by thermal convection, when air warmed near the Earth´s rises, and we can see that the resulting convection currents may spontaneously organize themselves in convection cells, either as long cylindrical rolls or in some cases forming a pattern that resembles s honeycomb when viewed from above.[19]

Based in the Rayleigh-Bénard convection, which will be discussed below, Lorenz published in 1963 a paper describing the numerical observed behaviour of solutions of a system of three first-order ordinary differential equations with simple nonlinearities:

$$\frac{dx}{dt} = -ax + ay$$

$$\frac{dy}{dt} = cx - y - xz$$

$$\frac{dz}{dt} = -bz + xy$$

where a, b, and c are constants. If we give these parameters the following values, *a* = 10, *b* = 8/3, and *c* = 28, are obtained the already classic Lorenz´s attractor in

[19] THOMPSON, J.M.T. and STEWART (1991): pp. 212-214.

which we can see without difficulty the wings of a butterfly.[20] Taking into account that in reality we are treating with a model of thermal convection, we are going to consider, as was announced, the essence of the Rayleigh-Bénard.

To explain briefly the Rayleigh-Bénard convection frequently is used a layer or liquid, e.g. water between two parallel plates. If the temperature of the bottom plate is the same as the top one the liquid will then tend towards an equilibrium that is asymptotically stable. But if there is a perturbation of the outside temperature, then the temperature of the bottom is increasing slightly yielding a flow of thermal energy conducted through the liquid. In principle there is no way to calculate the macroscopic effect of a microscopic perturbation of the initial conditions. If the temperature of the bottom plate was to be further increased, the structure would become more complex, and the turbulent flow become chaotic. Given that there is a density gradient between the top and the bottom plate, gravity acts trying to pull the cooler and denser liquid from the top to the bottom. We can see that this gravitational force is opposed by the viscous damping force in the fluid. The balance of these two forces is expressed by a non-dimensional parameter called the Rayleigh number. This and other experiments have demonstrated that chaos, or at least the route of chaos, exists in fluid dynamical systems.

The physic phenomenon of the Rayleigh-Bénard convection belongs to the area of fluid mechanics, but at the same time is in connection with the field of physics of the atmosphere in general, and with atmospheric flows in particular. In any case, we must say that the Rayleigh-Bénard convection is one of the most spectacular cases of chaotic behaviour, something to take in mind in this brief introduction to the possibilities of application of the theory of chaos to naval engineering.

In more general and enveloping terms the atmosphere and the ocean are large fluid masses that obey rather similar sets of physical laws. They both possess fields of motion that tend to be damped or attenuated by internal processes, and both fields of motion are driven, at least indirectly, by periodically varying external influences. With other words, each one is a very complicated forced dissipative dynamical system. Effectively, the atmosphere can be considered as an example of an intricate dynamical system whose irregularities are manifestations of chaos.[21]

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[20] FERNÁNDEZ DÍAZ, Andrés (2000): pp. 93-95.
We omit the figure of the attractor to save space, and for being widely known.

[21] LORENZ, Edward N. (1993): pp. 78-84.

The second coordinate to consider addresses the existence of chaos in fluid flows and, of course, also is concerned with the motion in atmosphere and with fluid mechanics principles and proprieties. Before entering into matter we must remember that our analysis, until the moment, has been realized from a macroscopic point of view, and that can be approached at a microscopic level, or what is more rigorous and precise, from a quantum perspective. In this respect we can say, for example, that the existence of a fluid causes a given particle to undergo an extremely irregular movement motion that is the Brownian motion. (Appendix C). We make this observation because, starting from the physical concept of deformation as the basis of the mathematical concept of topology, we replace the idea of a rigid body by its analysis in the continuum, admitting the inseparability in the space-time of the continent and the content, being the chaotic atmosphere and the flows of fluid the first, and the structure and behaviour of the ship the second.

In this second dimension or coordinate, when we approach the chaos in the fluid flows, we enter more fully into the branch of applied mechanics that studies the behaviour of the fluids, either at rest or in motion, and which constitutes what is called Fluid Mechanics and Hydraulics. We can consider, for instance, the nonlinear dynamics of the ocean flows, a subject that has been recently studied by researchers specialized in this field. In this sense it is well known that major currents like the Gulf Stream in the Atlantic and the Kuroshio in the Pacific and many other jet currents in different seas and oceans impact the waters through which they pass because they exchange their waters with specific temperature, salinity, chlorophyll concentration, etc., with ambient water. Such a fluid exchange may cause important effects on climate, biota, pollution propagation, giving rise, also to diverse types of processes in the ocean.[22]

Let us remind, before proceeding further, that the concept de advection refers to the transfer of heat due to the horizontal movement of a flow such as water, or if preferred, to the process of transporting a chemical substance by the movement on the medium containing it. From this definition we can express the advection equations as follows:

$$\frac{dr}{dt} = v\ (r\ , t)$$

[22] PRANTS, S.V.; BUDYANSKY, M.V. and ULEYSKY, M. Y. (2017): pp. 449-451.

This type of advection equation is generally nonlinear, given that the velocity v is a nonlinear function of the particle´s position r. That is to say, we deal with a nonlinear dynamic system whose solutions with a non-steady deterministic velocity field can be chaotic in the sense of exponential sensitivity to small variations in initial conditions and/or control parameters.[23]

Entering the third coordinate we find as the main theme the ship floatability and stability analysis and structural engineering dynamics. The term floatability refers to the flood weight gain reserve, depending on the number and position of watertight bulkheads and the waterline situation. For its part, stability is the characteristic by which the ship regains its position of equilibrium when accidental and diverse circumstances have inclined it, getting out of that irregular and dangerous situation. In this respect, it is easy to understand that stability depends mainly on the distribution of weights and the shapes and design of the ship.

Let us remember now that fluid mechanics tell us that there is a stable equilibrium of a submerged and floating body when the metacenter of center of thrust is above the center of gravity, the pair or forces being a restorative pair that takes the ship to its initial position. We say that there is an indifferent equilibrium if the metacenter and the center of gravity coincide. Finally, it can be affirmed that there is an unstable equilibrium when the metacenter is below the center of gravity. The pair of forces is then a capsizing pair.[24]

It should be clarified that the metacenter is referred to the crossing point of the thrust axis and the body´s float axis, the metacenter height being the distance between the metacenter and the center of gravity along the axis of flotation. Such a height constitutes a characteristic magnitude of the cross section of the body for a given weight, and the value an indicator of their stability.

We consider necessary to highlight that although the regulations developed by the International Maritime Organization (IMO) since its founding in 1959 have meant a clear increase in the safety of ships, accidents related to stability continue to occur, still complying with established criteria and standards. In this regard practice shows that even ships which satisfy all the existing rules are exposed to the risk of capsizing, which makes it necessary to know the existing trends for the improvement of the safety of ships. These trends are mainly oriented, may be,

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[23] A full and detailed analysis of this point can be seen in the paper written by Prants *et al.* cited in the previous note.

[24] MARTIN DOMINGO, Agustin (2019): pp. 15-18.
GILES, Ranald V.; EVETT, Jack B.; LIU, Cheng (1994): pp. 65-66.

towards shift from deterministic to probabilistic approach, uniting the knowledge of designers and direct experiences of ship crew members, and taking into account specific characteristic of certain ship types. But also is determinant to consider very seriously the influence of other factors missing from the existing rules.

Some of these other factors may be the capacity and knowledges of the ship crew members, the permanent and continuous information about the state and evolution of the weather, both in the atmosphere and in the sea, and the consequences of the nonlinear phenomena behaviour. This last point is really of great importance for the proposes of our analysis, given that ship rolling with monoharmonic excitation enables the insight in the series of nonlinear phenomena as shift of the natural frequency, symmetry breaking, resonant phenomena, subharmonic response, bifurcation, transition to chaos through cascade, fractal boundary, and chaotic attractor.

The nonlinear system could have a polyharmonic irregular response to monoharmonic excitation, which is called deterministic chaos. If we consider the ship rolling in sea waves, transfer to chaos i.e. fractal boundary of stability could be observed and the so called basin of erosion could be formed. In this regard we must not forget that estimation of the probability of ship capsizing for certain external influences and initial outer conditions are conducted using the method of basin erosion. [25]

Taking into account everything that has been seen in this subsection in which for reasons of space saving we have not entered into the detailed analysis of the underlying mathematical apparatus, we can say that in naval engineering, especially with regards to the phase of structures and design of ships, it is essential to consider the impact of nonlinearity and the existence of chaos that characterize the complex systems.

### f) Application in economics

Last but not least. Perhaps Economics is the scientific field in which chaos theory has been applied with more profusion. Under the general denomination of Chaotic Dynamics in Economics we can mention, among other matters, the income distribution, cycles and growth, general equilibrium, chaotic movement and regulation of demand, chaos and capital markets, and so on, there being an extensive literature on all these subjects. As an example, and given that it is

[25] BUČA, Pedišič and SENJANOVIČ, Ivo (2007): pp. 321-323.

possible to work with long time series, we have choose the evolution of the Madrid stock exchange during the lapse of time 2006-2013 that embraces the years of crisis that has its origin in the behaviour of financial markets. The analysis that was realized with the time series of IBEX 35 daily data of stock prices from January 2006 to December 2013 is integrated by 2019 data, and its time evolution appears in the figure nº 1 that we allow ourselves to reproduce here for further illustration.

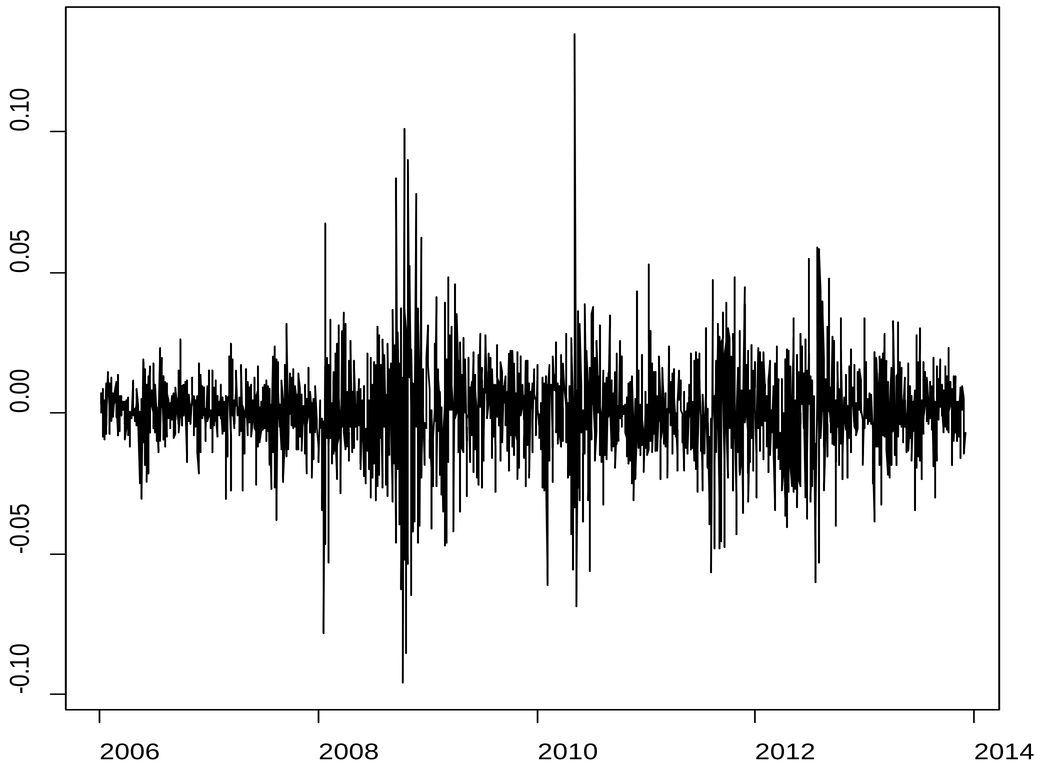


Figure 1: Time evolution of the IBEX 35 daily returns (2006-2013).

According to the statistical summary of data, the daily returns average for this period is very small, close to 0, while the daily volatility is around 1,7%. The smallest daily return is of -9,6%, whereas the largest is of 13,4%. The returns present a positive and small asymmetry, and a high kurtosis. These results are summarized in the Table 1. Likewise the hypothesis of normality of the data is rejected using the Jarque-Bera, and the presence of autocorrelation with the Ljung-Box test.

**Table .1**: Summary of statistical data

| | |
|---|---|
| Max | 13.484% |
| Min | -9.586% |
| Average | -0.005% |
| Standard Deviation | 1.653% |
| Asymmetry | 0.156 |
| Kurtosis | 8.823 |
| Jarque-Bera (p-value) | 0.000 |
| Ljung-Box (20 retards (p-value) | 0.000 |

After using the various tests for detecting chaos and considering all results obtained in our empirical analysis, we cannot conclude clearly that the IBEX 35 daily returns in the period of crisis be covered by a deterministic system. However the series could also be described as a nonlinear dynamic system with noise far from the equilibrium, submissive to bifurcations or changes in the dynamical of the system. The problem with this approach is that although the dynamical of the system it could be modelling correctly, could not obtain better predictions.[26]

Nevertheless it is necessary to emphasize that ethic, politic and institutional factors during the period analyzed can explain the irregularities and disequilibrium in the economy and in the international financial markets. The existence of corruption, the behaviour of the firms of rating with oligopoly power in the evaluation of the economic and financial policy of the different countries that work in the markets, and colluding with groups of brokers, are key elements in explaining the unusual and unconventional operation of capital markets in the years of the crisis.

All that can be interpreted as evidences of that there has been something more that hazard, or what is the same, that there has been a chaotic behaviour in the evolution of the series of values, and has been seen, some ability to predict.

It is important to stand out and to insist on the fact that with the application of these techniques of chaos, as we have seen in the example chosen, it is possible to distinguish between determinism or randomness when we are working with a time series. But we must point out that among statements that can be made relative to

[26] For a complete analysis see:
FERNÁNDEZ DÍAZ, Andrés; GRAU-CARLES, Pilar (2014): pp. 251-260.

complexity there is one according to wihich the systems are very often neither completely deterministic nor completely random, and exhibit both characteristics. The answer or solution can be found, as we know, in the results when we apply the different tests to detect nonlinearities and the existence of chaos in a given time series. In any case we are in conditions of obteining a better degree of information, and consequentely, a less degree of uncertainty about the phenomenum object of study.

When, as we have shown, it is found that a time series does not behave randomly due to the existence of chaos, ie, because underlies a causal relationship, it may be considered as a indication that the prediction is possible. All this despite the trend to identify chaos whith unpredictability, something that we reject, although it is necessary to recognized that reasonably accurate predictions would only be possible for very short time horizons.

We must not forget the great importance that has achieved in the last years the specific chapter of controlling chaos applied to the field of Economics (see 13 reference), finding a clear parallelism between the control of perturbations to cancel or to stabilize the chaotic behaviour, and the problem in economic policy of getting the necessary instruments or control variables for doing possible the fine tuning approach to reach the objectives established by the policy-maker. The development of this point within the applications of theory of chaos to economics constitutes certainly one of the most fructiferous research task at present.

To end, we want to remember in a broader sense, as we said earlier, that the chaos control constitues for the reasons already set out, a field of research of vital importance in the different areas of science, not only from the theoretical point of view, but also in their respective applications.

## APPENDIX A: METRIC SPACES AND TOPOLOGY

For any real numbers x and y, the Euclidean distance between them is the number

$$d_E(x, y) = |x - y| \in \mathbb{R}^+$$

Generalizing this idea, a set having a distance measure or metric, defined for each pair of elements is called a *metric space*.

Let S denote such a set. Then a metric is a mapping $d$: S x S $\mapsto$ $\mathbb{R}^+$ having the properties:

1. $d(y, x) = d(x,y)$
2. $d(x, y) = 0$ iff if $x=y$
3. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

It is possible now define a metric space (S, *d*) as a set S paired with metric *d*, such that the precedent conditions hold for each pair of elements.[27]A metric space (S, *d*) it said is complete if every Cauchy sequence converges at an element x, that is, there is an element of the space that is the limit of the sequence.

Is advisable to clarify that the terms mapping, transformation, etc, are synonyms for function, but an extra usage is *functional*, which refers to the case where the domain is a space whose elements are themselves functions, with $\mathbb{R}$ as co-domain. An example may be found in the text of this article when we deal with the application of optimal control theory to economic policy, in which we introduce the condition of maximize a functional for getting the optimal evolution of economy.

Finally, we must point out that metric spaces form a subclass of a larger class of mathematical objects called topological spaces. These do not have a distance defined upon them, but the concept of open set, neighborhood, and continuous mapping are still well defined. Even though only metric spaces are encountered in the sequel, much of the reasoning is essentially topological in character.

Poincaré published a large number of papers on the so-called automorphic and fuchsia functions, on differential equations and topology. In this last branch his work

[27] For a more developed analysis see:
FAISANT, Alain (1977): pp. 157-182.

was really outstanding, what is important taking into account its relation, subsequently, with chaos and complexity.

NOTE: Something about Topology.

The base of the classic concept of movement rested in the distinction between the underlying metric vessel and their variable physical content, that is, between empty and full. The indifferent geometric vessel does not exist in the general relativity theory. Thus the material is converted to local deformation of space-time medium. More precisely, what was considered as a material body is nothing more than a center of this deformation whose movement is accompanied by the concomitant movement of the whole.

This physical concept of deformation it is the underlying concept of topology. Indeed, and as discussed below, the deformation is a fundamental property of topological bodies. We can star from the concept of "structure" and their four types: algebraic, of order, topological and complex. The last one, that is, the topological structure, refers to neighborly relations, limit and continuity.

Topology or "Analysis Situs", that so is also denominating, is an extension within the field of geometry. Beyond the metric geometry and the projective one, inspired in the notions of distance and straight line, respectively, appears a third geometry, the "qualitative", in which don´t works the quantity: we are speaking of the Topology. In it two figures are equivalent always that been possible to pass of one to another due to a continuous deformation.

Let us suppose that we have a rubber ball. If we go by the pressure deforming without breaking or altering their nature, will have properties of primitive figure that will not change: these are the topological properties. Its development has allows the treatment and systematization of issues as the theory of dimension, the differential geometry of curves and surfaces, the Poincare´s polygon, etc.

To conclude, it is relevant to point out that since many years the topology has been applied to Economics to solve important and complex problems, mainly in the specific field of the General Equilibrium, without forget others areas of our science.

## APPENDIX B: MARTINGALES

We denote by

$$H_{t-1} = h\,(x_{t-1}, x_{t-2}, \ldots) \qquad (1)$$

an information set content in a stochastic process until the time t-1.

Then we can say that a stochastic process is a martingale if comply the following conditions:

1. There is a μ (t )⟶ ∀t, where

$$\mu\,(t) = E\,[X(t)] = \int_{-\infty}^{\infty} x\, dF(x,t) \qquad (2)$$

2. Is verified that

$$\mu(t) = E\,[X(t) / H_{t-1}] = X(t-1) \longrightarrow \forall t \qquad (3)$$

what implies for any $\tau \geq 0$

$$E\,[X(t+\tau) / H_{t-1}] = X(t-1) \qquad (4)$$

We can now define the following concepts:

Submartingale…………E[ X(t)/ $H_{t-1}$ ] $\geq X(t-1)$

Supermartingale............E[(X(t)/ $H_{t-1}$] $\leq X(t-1)$ (5)

Semimartingale………..E[ X(t)/ $H_{t-1}$ ] $> X(t-1)$

that is, the best predictor of a stochastic process starting from t is always the preceding observation X(t -1).

It is important to remember that is possible to express or study martingale in relation to random walk that, as we know, is one of the simplest random processes. We see without stop this possibility.

Let us begin with the definition of random walk:

$$Y_t = Y_{t-1} + \varepsilon \qquad (6)$$

where ε is a random variable which zero mean and a constant variance, and where there is zero correlation between observations.

With drift element that actually refer to a time trend the equation would be

$$Y_t = Y_{t-1} + \alpha + \varepsilon \tag{7}$$

and depending of the value of $\alpha$ we should have the following:

If $\alpha = 0$, we have a random walk

If $\alpha > 0$, we have a submartingale (8)

If $\alpha < 0$, we have a supramartingale

Martingale is a more general stochastic process than a random because need not have constant variance. The term drift is used to represent the positive or negative trend in the time series of the stochastic variable.

Finally, is important to emphasize that a martingale is always defined with respect to some information set, and with respect to some probability measure. If we change the information content and/or the probabilities associated with the process, the process under consideration may cease to be a martingale. The opposite is also true. Effectively, given a process $X_t$, which does not behave like a martingale, we may be able to modify the relevant probability P and convert $X_t$ into a martingale.

Note: White noise.

A time series is said to be "white noise" if the underlying variable has zero mean, a constant variance and zero correlation between successive observations.

## APPENDIX C: WIENER PROCESS AND BROWNIAN MOTION

A **Wiener process** $W_t$ relative to a family of information sets $\{I_t\}$, is a stochastic process such that

1. The pair $I_t$, $W_t$ is a square integrable martingale with $W_0 = 0$ and

$$E\,[(W_t - W_s)^2] = t - s; \qquad s \leq t \tag{1}$$

2. The trajectories of $W_t$ are continuous over *t.*

Starting from this definition we have the following properties of a Wiener process:

1ª.-$W_t$ has uncorrelated increments because it is a martingale, and because every martingale has unpredictable increments.

2ª.-$W_t$ has zero mean because it starts at zero, and the mean of every increment equals zero.

3ª.-$W_t$ has variance *t.*

4ª.-The process is continuous in the sense that in infinitesimal intervals, the movements of $W_t$ are infinitesimal.

We must know that the Wiener process is the natural model for an asset price that has unpredictable increments but nevertheless moves over time continuously.

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Now we are going to give the definition of **Brownian motion**:

A random process $B$, $t \in [0, T]$ is a Brownian motion if:

1º.-The process begins at zero, $B_0 = 0$.

*2º.-B* has stationary, independent increments.

3º.-The $B_t$ is continuous in t.

4º.-The increments $B_t - B_s$, have a normal distribution with mean zero and variance $|t - s|$:

$$(B_t - B_s) \sim N(0, |t - s|). \tag{2}$$

This definition is, in many ways, similar to that of the Wiener process. Then we can establish as a theorem that any Wiener process $W_t$ relative to a family $I_t$ is a Brownian motion process.[28] However there is an important difference consisting in the fact that $W_t$ was assumed to be a martingale, while no such statement is made about $B_t$, that is posited has a normal distribution. In any case, this theorem is very explicit, given that we can use the terms Wiener process and Brownian motion interchangeably.

The Brownian motion constitutes one of the theories that have had more success in the study of the capital markets, and in its behaviour we find a clear parallelism between the displacement of a particle and the time series of assets returns, as the later seems to evolve without a fixed pattern and seemingly random. In this regard let us not forget that in 1923 Wiener introduced the concept of random Brownian motion function.

If we consider a normalized Gaussian random process $\{ \xi \}$, the increase in the position of a Brownian particle is:

$$X(t) - X(0) \longrightarrow \xi \, | t - t_0 |^{H}, \text{ where } H=0,5 \tag{3}$$

that is, if we want to know the position of a particle in a given moment t, knowing the position in a reference point $t_0$, a random number $\xi$ that follow a Gaussian distribution is chosen, multiplied by $|t - t_0|^{0,5}$, and this result is added to the position in $t_0$, that is $X(t_0)$.

A generalization of this function is the fractional Brownian motion introduced by Mandrelbrot and Van Ness (1968). In the particular case of H=0,5 the random function corresponds to the ordinary Brownian motion.

[28] For a more deep analysis see:
NEFTCI, S. N. (1996): pp. 143-167.

## APENDIX D: DIFFUSION AND TELEGRAPH EQUATIONS[29]

The prototype diffusion equation, also called the heat equation, can be expressed in this way:

$$\frac{\partial \psi}{\partial t} = D\nabla^2 \psi \qquad (1)$$

that is,

$$\frac{\partial \psi}{\partial t} = D\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] \qquad (2)$$

where *D* is the diffusion coefficient.

In a one dimension it would

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} \qquad (3)$$

whose solution would be

$$\psi(x,t) = \frac{1}{2(\pi Dt)^{1/2}} e^{-x^2/4Dt} \quad \rightarrow \quad t > 0 \qquad (4)$$

Let us take a more simplify version of (4), where the coefficient D is implicit[30]

$$\psi(x,t) = t^{-1/2} \cdot e^{-x^2/2t} \qquad (5)$$

For different values of *t*, and varying x between –6 y 6, could be obtained the graphic representation of the diffusion process, as we can see in the figure 1:

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[29] There is a more extensive version of this last Appendix in our book: FERNÁNDEZ DÍAZ, Andrés (2000): pp. 237-250.

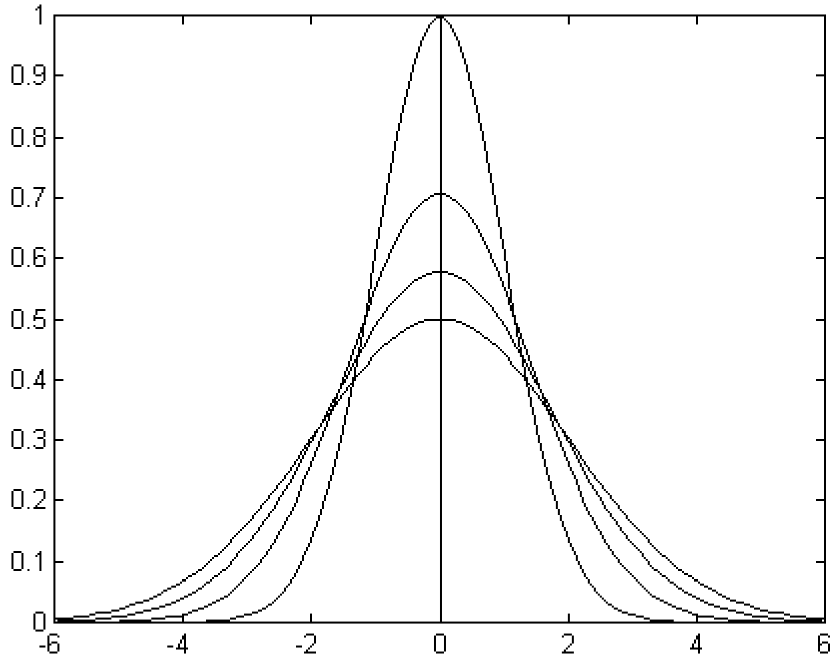


*Figure 1*

Note that the curve is having fewer slopes as it increases the value of t, remaining constant the surface regardless of that undergoes a process of diffusion. Of course, the analysis could be extended to the case of two and three dimensions.

Related to the diffusion equation we can to consider the so-call "telegraph equation", which takes the form

$$\frac{1}{a}\frac{\partial^2\psi}{\partial t^2}+\frac{\lambda}{a}\frac{\partial\psi}{\partial t}=D\frac{\partial^2\psi}{\partial x^2}+\beta\frac{\partial\psi}{\partial x} \qquad (6)$$

where $D$ and $\beta$ are adequately defined.[31]. The equation shows that an initial impulse propagates as a wave, and subsequently by a diffusion process. So, the sequence would be impulse ⇒ wave ⇒ diffusion ⇒ impulse...

Both the diffusion equation as Telegraph has a structure of great interest to explain some economic phenomena. In effect, this type of equations lead to power laws

[31] Note that in (6) are included the components of the wave equation, being this one, as we know, and for only one dimension, $\frac{\partial^2\psi}{\partial t^2}=a^2\frac{\partial^2\psi}{\partial x^2}$ .

and to behaviour and scenarios of avalanches and self-organized critically that may be illustrative to study outstanding and complex fields of Economics.

As an example we will consider an interpretation of Black-Scholes model to market options by studying certain stochastic processes.

Let us start of S, the price of an asset that we can consider as a random variable that follows a Wiener process, expressing in the following way a change in a little interval of time:

$$dS = \varepsilon\sqrt{dt} \qquad (1)$$

where $\varepsilon$ is a standard normal variable, being therefore *dS* normally distributed, with mean zero and standard deviation $\sqrt{dt}$ .

Given that the different securities or bonds have volatility, multiply the second member of the equation (1) by $\sigma$, that is the annualized standard deviation of *dS*. It would then have:

$$dS = \sigma\varepsilon\sqrt{dt} \qquad (2)$$

But as it makes no sense a null change of price of asset, it would be necessary to add a parameter to represent the positive or negative trend in the time series of the stochastic variable (drift parameter).

The (2) would remain so:

$$dS = \alpha dt + \sigma\ \varepsilon\ \sqrt{dt} \qquad (3)$$

indicating with $\alpha\ dt$ the mean or the expected value of the rate of return.

A generalization of a Wiener process is constituted by the Ito process, in which so the expected return ( $\alpha$ ) as ( $\sigma$ ) are dependent on the underlying variables *S* and *t*.

The equation that would express the Ito process would be

$$dS = \alpha(S,t) + \sigma(S,t)\ \varepsilon\sqrt{dt} \qquad (4)$$

and more concretely

$$dS = \alpha S dt + \sigma S\varepsilon\sqrt{dt} \qquad (5)$$

With an Ito process we could value derivatives as forwards, futures and options, taking into account that any variable that is function of others that follow an Ito process also follows such one.

In this manner, if *W* is value of a financial product derived of other whose price is S, we can write:

$$dW(S,t) = \left[\frac{\partial W}{\partial S} aS + \frac{\partial W}{\partial t} + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial W}{\partial S}\sigma S \varepsilon\sqrt{dt} \qquad (6)$$

reflecting the formulation content in the bracket what we have denominate "drift rate".

An expression of this type, well known and used in the financial analysis, is the Black-Scholes equation

$$\frac{\partial W}{\partial t} + rS\frac{\partial W}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} = rW \qquad (7)$$

where r indicates, in certain per one, the rate of official basic interest, or without risk.

Removing, the (7) would remain so:

$$\frac{\partial W}{\partial t} = rW - rS\frac{\partial W}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} \qquad (8)$$

Note that in the left side of the equation remains, in the limit, what in the terminology of the financial markets is called the Theta of a portfolio of options or of an option, $\Theta$, that is the rate of variation of the value of the portfolio or of the very option over time.

Likewise, and following with the habitual terminology, in the right side we find what is known as the Delta of an option, $\Delta$, which it is defined as the rate of change of it price with respect to the price of the asset underlying, or what is the same, the slope of the curve that relate the change of the price of the option *($\Delta W$)* with that one that feels the price of the stock that serves as base *($\Delta S$)*; in the limit we should have then the expression $\frac{\partial W}{\partial S}$ , or the rate of the change $\frac{\partial^2 W}{\partial S^2}$.

The equation (8) would be equivalent to the following:

$$\Theta = \mathrm{rW} - \mathrm{rS}\Delta - \frac{1}{2}\sigma^2 S^2 \frac{\partial(\Delta)}{\partial S} \qquad (9)$$

Given, on the other hand, that the rate of change of the Delta of the option with respect to the price of the underlying asset may be defined, in his turn, as the Gamma ( $\Gamma$) of a portfolio of options, or of an option over an underlying asset, we can arrive, finally, to the formula

$$\Theta = \mathrm{rW} - \mathrm{rS}\Delta - \frac{1}{2}\sigma^2 S^2 \Gamma \qquad (10)$$

However, and in the framework of our analysis, it is preferable to return to the formalization of the (8), thus avoiding any confusion in handling the $\Delta$ in the financial sense, on the one hand, and the laplacian of a dimension that is part of the equation, explicit or implicitly, on the other.

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